Quantum mechanics II, Chapter 2 : Entanglement (Part 1)

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Sometimes observation kills.

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Problem 1 : Bell States

Consider two spin-1/2 particles A and B with :

$$|\psi(A)\rangle = c_0^{(A)}|0\rangle_A + c_1^{(A)}|1\rangle_A$$

and

$$|\psi(B)\rangle = c_0^{(B)}|0\rangle_B + c_1^{(B)}|1\rangle_B.$$

The states $|0\rangle$ and $|1\rangle$ are the eigenstates of the $\hat{S}_z = \frac{\hbar}{2}\sigma_z$ operator with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

1. Write down the four possible basis states for the composite system $|\psi(A)\rangle \otimes |\psi(B)\rangle$ in terms of the basis vectors $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$.
2. The Bell states are two-particle states given by :

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\right)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right).$$

   Show that these four Bell states form an orthonormal basis of the Hilbert space of the two spins $H = H_A \otimes H_B$.
3. Are the four Bell states separable ?
4. We now consider the Bell state $|\Psi^+\rangle$.
   (a) What is the probability of measuring $-\hbar/2$ when measuring the spin $\hat{S}_z^{(B)}$ of particle B ?
   (b) Let's assume now that the measurement of the spin B gives us $-\hbar/2$. What is the probability of subsequently measuring $+\hbar/2$ for the spin A ?
5. We now consider $|\theta\rangle = \frac{1}{2}|\Phi^+\rangle + \frac{\sqrt{3}}{2}|\Psi^-\rangle$. Is this state correctly normalized ? What is the probability of obtaining $\pm\hbar/2$ upon measuring $\hat{S}_z^{(A)}$ on the first spin ?

6. What is the probability of obtaining $\pm\hbar/2$ upon measuring $\hat{S}_x^{(A)}$ (the x-axis spin of the first particle) for all of the four Bell states ?

Problem 2 : Composite system of two spin-1/2 particles

Consider the following Hamiltonian operator for two spin-1/2 particles :

$$\hat{H} = \mu_x \hat{S}_x^{(A)} \otimes \hat{S}_x^{(B)} + \mu_y \hat{S}_y^{(A)} \otimes \hat{S}_y^{(B)}$$

where $\hat{S}_x^{(A)}$ and $\hat{S}_y^{(A)}$ are the spin operators for the first spin and $\hat{S}_x^{(B)}$ and $\hat{S}_y^{(B)}$ are the spin operators for the second spin.

1. What are the conditions on the coefficients μ_x and μ_y such that $\hat{H}$ is a valid observable ?
2. Write down the matrix elements of $\hat{H}$ in the basis of $\hat{S}_z$ (i.e. $\{|0\rangle, |1\rangle\}^{\otimes 2}$).
3. Diagonalize the Hamiltonian in this basis and find its eigenvalues and the corresponding eigenvectors.
4. Are the eigenvectors of the Hamiltonian separable between the two particles ?